STA 447/2006 Lecture 6

Recall OST.

e.g. Gambler's ruin.

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w.p. } p \\ X_n - 1 & \text{w.p. } (1-p). \end{cases}$$

$$T = \inf\{ n \geq 0 : X_n = 0 \text{ or } c \}.$$

Question : $\mathbb{E}[T]$?

- Sym case $p = \frac{1}{2}$.

$(X_n)_{n\geq 0}$ is an MG

Idea: construct another MG that involves n.

$$Y_n = X_n^2 - n$$

$$\mathbb{E}[Y_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\left[(X_n + \varepsilon_{n+1})^2 - (n+1) \,\middle|\, \mathcal{F}_n\right]$$

$$= X_n^2 - n = Y_n.$$

Suppose under ?? condition.

we could apply OST.

$$\mathbb{E}[Y_T] = \boxed{\mathbb{E}[X_T^2]} - \mathbb{E}[T]$$

$$\|$$

$$\mathbb{E}[Y_0] = a^2$$

(assuming $X_0 = a$)

$$= \mathbb{P}(X_T = c) \cdot c^2 + \mathbb{P}(X_T = 0) \cdot 0^2$$

$$= a \cdot c$$

Therefore,

$$\mathbb{E}[T] = ac - a^2 = a \cdot (c - a).$$

Still need to verify condition for OST.

$$\lim_{n\to+\infty} \mathbb{E}\left[|Y_n| \cdot \mathbb{1}_{T>n}\right] \overset{?}{=} 0.$$

($Y_n = X_n^2 - n$ not bounded up to T.)

$$\mathbb{E}\left[|Y_n| \mathbb{1}_{T>n}\right] \leq \mathbb{E}\left[(X_n^2+n)\mathbb{1}_{T>n}\right]$$

$$\leq \underbrace{c^2 \cdot \mathbb{P}(T>n)} + \underbrace{n \cdot \mathbb{P}(T>n)},$$

$\lim_{n\to+\infty} \mathbb{P}(T>n) = 0$

because $\mathbb{P}(T<+\infty)=1$.

Need to control tail/moments of T.

$$n \cdot \mathbb{P}(T>n) \leq \mathbb{E}\left[T \cdot \mathbb{1}_{T>n}\right]$$

$\mathbb{1}_{T>n} \to 0$ a.s. (since T is finite)

In order to apply DCT

We need to show $\mathbb{E}[T] < +\infty$.

From practice question / midterm 1.

For finite state space MC, irreducible

hitting time of any state

has exponential tail, i.e.

$\exists\, C > 0,\ \rho < 1$. s.t.

$$\mathbb{P}_i(T_j > n) \le C \cdot \rho^n, \quad (\forall i, j \in S)$$

In order to apply this result we consider

$(\tilde{X}_n)_{n \ge 0}$ making transitions

$$\begin{cases} \text{same as } X_n & \text{when } 0 < \tilde{X}_n < c \\ \tilde{X}_n + 1 & \text{when } \tilde{X}_n = 0 \\ \tilde{X}_n - 1 & \text{when } \tilde{X}_n = c. \end{cases}$$

$(\tilde{X}_n)$ irreducible $\quad X_n = \tilde{X}_n$ up to time T.

So we have $\mathbb{P}(T > n) \leq C' \rho^n$ for the process $(X_n)_{n\geq 0}$, and OST is applicable.

We also verify $\mathbb{E}[|Y_T|] < +\infty$

$$\mathbb{E}[|Y_T|] \leq \mathbb{E}[X_T^2] + \mathbb{E}[T] < +\infty$$

[Note: OST only tells us how to compute $\mathbb{E}(T)$ when it's finite, we need to verify it's finite separately].

· Asymmetric case,

$$Y_n = X_n - (2p-1)n \quad \text{is MG.}$$

$$\mathbb{E}\left[|Y_n| \cdot \mathbb{1}_{T>n}\right]$$

$$\leq (c+(2p-1)n) \cdot \mathbb{P}(T>n) \longrightarrow 0$$

$$\mathbb{E}[|Y_T|] \leq \mathbb{E}[c+(2p-1)\cdot T] < +\infty.$$

So by OST,

$$0 = \mathbb{E}[Y_0] = \mathbb{E}[Y_T] = \mathbb{E}[X_T] - (2p-1)\mathbb{E}[T].$$

$$= \underbrace{c \cdot \mathbb{P}(X_T = c)}_{\text{Computable from last lecture}} + \cancel{0 \cdot \mathbb{P}(X_T = 0)} - (2p-1)\mathbb{E}[T]$$

Using polynomial / exponential MGs, we can compute moments / chf of T.

Recall the limit condition in OST.

$$\mathbb{E}\left[|X_n| \cdot \mathbb{1}_{A_n}\right] \to 0$$

for $A_n = \{T > n\}$

We know $\mathbb{P}(A_n) \to 0$.

(by DCT, we know that

$$\mathbb{E}\left[|X| \cdot \mathbb{1}_{A_n}\right] \to 0$$

as long as $\mathbb{E}[|X|] < +\infty$).

Analogous notion for sequence:

uniform integrability.

c.f. integrable r.v. X

$$\lim_{K \to +\infty} \mathbb{E}\left[|X| \cdot \mathbb{1}_{|X| > K}\right] = 0.$$

$\forall \varepsilon > 0, \exists K > 0$ s.t. $\mathbb{E}[|X| \cdot \mathbb{1}_{|X|>K}] \leq \varepsilon$.

De A collection of r.v.'s $(X_n)_{n \geq 0}$

$\mathbb{E}[|X_n|] < +\infty$ $(\forall n)$.

We call them uniformly integrable if

$\forall \varepsilon > 0, \exists K < +\infty$ s.t.

$$\forall n, \quad \mathbb{E}[|X_n| \cdot \mathbb{1}_{|X_n|>K}] < \varepsilon$$

(Same K that works for all n).

Thm. Let $(X_n)_{n \geq 0}$ be uniformly integrable martingale. Stopping time T $\left|\begin{array}{l} T < +\infty \text{ (a.s.)} \\ \mathbb{E}[|X_T|] < +\infty. \end{array}\right.$

then we have

$$\mathbb{E}[|X_n| \mathbb{1}_{T > n}] \to 0$$

and therefore $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

Proof. Let $A_n = \{T > n\}$.

$$\mathbb{E}[|X_n| \cdot \mathbb{1}_{A_n}]$$

$$= \mathbb{E}[|X_n| \cdot \mathbb{1}_{A_n \cap \{|X_n| > K\}}] + \mathbb{E}[|X_n| \mathbb{1}_{A_n \cap \{|X_n| \le K\}}]$$

$$\le \mathbb{E}[|X_n| \cdot \mathbb{1}_{|X_n| > K}] + K \cdot \mathbb{P}(A_n).$$

K indp of n, from u.i.

Fix $\varepsilon > 0$, let $K = K(\varepsilon)$ be the K from def of u.i.

We have

$$\mathbb{E}[|X_n| \mathbb{1}_{A_n}] \le \varepsilon + K \cdot \mathbb{P}(A_n). \quad (\forall n \in \mathbb{N}_+)$$

$(\mathbb{P}(A_n) = \mathbb{P}(T > n) \to 0)$

Take $n \to +\infty$.

$\forall \varepsilon > 0$, $\lim_{n \to +\infty} \mathbb{E}[|X_n| \cdot \mathbb{1}_{A_n}] \le \varepsilon$.

So $\lim_{n \to +\infty} \mathbb{E}[|X_n| \cdot \mathbb{1}_{A_n}] = 0$.

e.g. Suppose that $\exists\, C < +\infty$

s.t. $\mathbb{E}\left[|X_n|^2\right] \leq C$, $\quad \forall n.$

then $(X_n)_{n \geq 0}$ is u.i.

Proof. $\mathbb{E}\left[|X_n| \mathbb{1}_{|X_n| \geq K}\right]$

$$\leq \sqrt{\mathbb{E}[X_n^2]} \cdot \sqrt{\mathbb{P}(|X_n| \geq K)}$$

(Cauchy-Schwarz)

$$\leq \sqrt{C \cdot \mathbb{P}(|X_n| \geq K)}.$$

(Markov) $\leq \sqrt{C \cdot \dfrac{\mathbb{E}[X_n^2]}{K^2}}$

$$\leq \frac{C}{K}$$

So for $\varepsilon > 0$, we can take $K = \dfrac{C}{\varepsilon}$.

e.g. $X_n = \sum_{j=1}^{n} \frac{1}{j} Z_j$

where $Z_j \overset{iid}{\sim} \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2. \end{cases}$

$$\mathbb{E}[X_n^2] = \sum_{j=1}^{n} \frac{1}{j^2} \leq \sum_{j=1}^{+\infty} 1/j^2 = \frac{\pi^2}{6} < +\infty.$$

So $(X_n)_{n \geq 1}$ is u.i.

Note: only uniformly bounded first moment is not enough for u.i.

In general, if we have

$$\sup_{n \geq 1} \mathbb{E}\left[|X_n|^{1+\varepsilon}\right] \leq C < +\infty \quad (\text{for some } \varepsilon > 0).$$

This implies u.i.

Non-example. $(Z_n)_{n \geq 0}$ 1-D symmetric SRW.

T is hitting time of -1.

$$X_n := 1 + Z_{\min\{T,n\}} \quad \left(= \begin{cases} 1+Z_n, & (n \le T) \\ 1+Z_T = 0, & (n > T) \end{cases} \right)$$

$$\mathbb{E}[|X_n|] = \mathbb{E}[X_n]$$
$$= 1 + \underbrace{\mathbb{E}[Z_{\min\{T,n\}}]}_{=0 \text{ by optional stopping lemma}}$$
$$= 1 < +\infty \quad (\forall n).$$

We'll see in next lecture that $(X_n)_{n \ge 0}$ is not u.i.

Another OST-type result

Wald's identity.

$$X_n = \sum_{j=1}^{n} Z_j$$

where Z_j's are i.i.d. $\mathbb{E}[|Z_j|] < +\infty$.

$$m := \mathbb{E}[Z_1].$$

$$Y_n = X_n - mn \quad (\forall n) \text{ is MG.}$$

Thm (Wald): If T is a stopping time satisfying $\mathbb{E}[T] < +\infty$, we have

$$\mathbb{E}[X_T] = m \cdot \mathbb{E}[T].$$

(cf. biased RW, calculation of $\mathbb{E}[T]$).

Proof. (Notation: $a \wedge b = \min(a, b)$).

By O.S. Lemma, $\forall n$,

$$\mathbb{E}[X_{n \wedge T}] = m \cdot \mathbb{E}[n \wedge T]$$

by DCT, $\mathbb{E}[n \wedge T] \to \mathbb{E}[T]$.

So it suffices to study

$$\left| \mathbb{E}[X_{n \wedge T}] - \mathbb{E}[X_T] \right|$$

$$\leq \mathbb{E}\left[|X_{n\wedge T} - X_T| \right]$$

$$= \mathbb{E}\left[\mathbb{1}_{\{n < T\}} \left| \sum_{m=n+1}^{T} Z_m \right| \right]$$

$$\leq \mathbb{E}\left[\sum_{m=n+1}^{T} |Z_m| \cdot \mathbb{1}_{n<T} \right]$$

$$= \mathbb{E}\left[\sum_{m=n+1}^{+\infty} |Z_m| \cdot \mathbb{1}_{T \geq m} \right]$$

$$\underset{\text{(Fubini-Tonelli)}}{=} \sum_{m=n+1}^{+\infty} \mathbb{E}\left[|Z_m| \cdot \mathbb{1}_{T \geq m} \right].$$

Key observation:

$$\{T \geq m\} = \{T \leq m-1\}^c$$

is determined by $Z_1, Z_2, \ldots, Z_{m-1}$

So Z_m is indp with $\mathbb{1}_{T \geq m}$.

$$\mathbb{E}\left[|Z_m| \mathbb{1}_{T \geq m}\right] = \mathbb{E}\left[|Z_m|\right] \cdot \mathbb{P}(T \geq m).$$

So $\left| \mathbb{E}[X_{n \wedge T}] - \mathbb{E}[X_T] \right|$

$$\leq \sum_{m=n+1}^{+\infty} \mathbb{E}\left[|Z_m| \cdot \mathbb{1}_{T \geq m}\right]$$

$$= \mathbb{E}\left[|Z_1|\right] \cdot \sum_{m=n+1}^{+\infty} \mathbb{P}(T \geq m).$$

$$\sum_{m=1}^{+\infty} \mathbb{P}(T \geq m) = \mathbb{E}[T] < +\infty.$$

So $\lim_{n \to +\infty} \sum_{m=n+1}^{+\infty} \mathbb{P}(T \geq m) = 0.$

this completes the proof of Wald's thm.

Rmk: by using structures in iid sum.

Wald's thm proves an OST result under mild conditions

Another application of OST: Doob's maximal ineq.

Recall Markov ineq.

Let X be a non-negative r.v.

$$\forall \lambda > 0, \quad \mathbb{P}(X \geq \lambda) \leq \frac{\mathbb{E}[X]}{\lambda}.$$

For a sequence $(X_n)_{n \geq 0}$

Assuming $(X_n)_{n\geq 0}$ is a non-negative MG.

$$\mathbb{P}(X_n \geq \lambda) \leq \frac{\mathbb{E}[X_n]}{\lambda} = \frac{\mathbb{E}[X_0]}{\lambda}.$$

How about

$$\mathbb{P}\left(\max_{1 \leq t \leq n} X_t \geq \lambda\right).$$

Thm (Doob)

$$\mathbb{P}\left(\max_{1 \le t \le n} X_t \ge \lambda\right) \le \frac{\mathbb{E}[X_0]}{\lambda}.$$

Proof. (argmax is not a stopping time, but first time $\ge \lambda$ is)

$$T := \inf\{t \ge 1 : X_t \ge \lambda\}.$$

$$\mathbb{P}\left(\max_{1 \le t \le n} X_t \ge \lambda\right) = \mathbb{P}(T \le n)$$

By optional stopping lemma.

$$\mathbb{E}[X_{T \wedge n}] = \mathbb{E}[X_0].$$

$$\begin{aligned} &\| \\ &\mathbb{P}(T > n) \cdot \mathbb{E}[X_{T \wedge n} \mid T > n] \\ &\quad + \mathbb{P}(T \le n) \cdot \mathbb{E}[X_{T \wedge n} \mid T \le n]. \end{aligned}$$

$$\geq \mathbb{P}(T \leq n) \cdot \mathbb{E}[X_T \mid T \leq n].$$

On the event $T \leq n$ we have

$$X_T \geq \lambda \quad a.s.$$

therefore, $\mathbb{E}[X_T \mid T \leq n] \geq \lambda.$

So we have

$$\mathbb{E}[X_0] = \mathbb{E}[X_{T \wedge n}]$$

$$\geq \lambda \cdot \mathbb{P}(T \leq n).$$

So $\mathbb{P}(T \leq n) \leq \dfrac{\mathbb{E}[X_0]}{\lambda}$

which proves the max. ineq.

MG convergence.

Note :

MC convergence $\quad X_n \xrightarrow{d} \pi.$

MG ---- $\quad X_n \xrightarrow{a.s.} X_\infty$

i.e. $\quad \mathbb{P}(X_n \to X_\infty) = 1.$

e.g. Gambler's ruin.

$(X_n)_{n\geq 0} \qquad X_0 = a.$

$T =$ hitting time of $\{0, c\}$

$(X_{n\wedge T})_{n\geq 0}$ is a MG.

From previous discussion, we know

$$\mathbb{P}\left(\lim_{n\to+\infty} X_{n\wedge T} = X_T\right) = 1.$$

The limiting r.v. is $\begin{cases} c & \text{w.p. } a/c \\ 0 & \text{w.p. } 1 - a/c. \end{cases}$

eg. $\sum_{j=1}^{+\infty} j^{-1} Z_j$ where $Z_j \overset{iid}{\sim} \begin{cases} +1 & \text{w.p. } 1/2 \\ -1 & \text{w.p. } 1/2. \end{cases}$

If $Z_1 = Z_2 = \cdots = Z_n = \cdots = 1$. diverge

But this happens w.p. 0.

We'll show $(X_n)_{n\geq 0}$ converges a.s.

Thm (MG convergence).

Suppose $(X_n)_{n\geq 0}$ is MG.

If $\exists\, C < +\infty$, s.t. $\mathbb{E}[|X_n|] \leq C$, $(\forall n)$

(strictly weaker than u.i.)

then $\exists$ a limiting r.v. X_∞

s.t. $\mathbb{P}(X_n \longrightarrow X_\infty) = 1$.

eg. random harmonic series Converges.

Cor. Suppose $(X_n)_{n\geq 0}$ M.G.

$X_n \geq -a$ a.s. $(\forall n)$

for some constant a.

Then $X_n \longrightarrow X_\infty$ a.s.

Proof of corollary.

$$\mathbb{E}[|X_n|] \leq |a| + \mathbb{E}[|X_n + a|].$$

$$= |a| + \mathbb{E}[X_n + a]$$

$$\leq 2|a| + \mathbb{E}[|X_0|] < +\infty$$