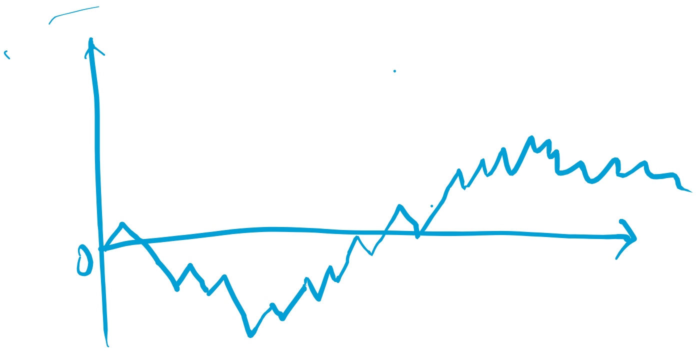
STA 447/2006 Lecture 8

Idea: 1-D SRW. $X_n = \sum_{i=1}^{n} \varepsilon_i$ where $\varepsilon_i \begin{cases} +1 & \text{w.p. } \frac{1}{2} \\ -1 & \text{w.p. } \frac{1}{2}. \end{cases}$



Scaling limit. $\frac{1}{\sqrt{n}} X_n \xrightarrow{d} N(0,1)$.

Joint distribution of $(X_{n_1}, X_{n_2}, \ldots, X_{n_k})$

$n_i = \lfloor n \cdot t_i \rfloor$ where $n \to +\infty$

$t_1, t_2, \ldots, t_k$ fixed.

Assuming WLOG that $0 \le t_1 < t_2 < \cdots < t_k < +\infty$.

$$\frac{1}{\sqrt{n}} \begin{bmatrix} X_{n_1} \\ X_{n_2} - X_{n_1} \\ \vdots \\ X_{n_k} - X_{n_{k-1}} \end{bmatrix} \xrightarrow{d} N\left(0, \begin{bmatrix} t_1 & & 0 \\ & t_2 - t_1 & \\ & & \ddots \\ 0 & & & t_k - t_{k-1} \end{bmatrix}\right)$$

Take markers $t_1, t_2, \ldots, t_k$ denser

we have the joint distribution

$$\left(\frac{1}{\sqrt{n}} \cdot X_{\lfloor nt \rfloor} : 0 \le t \le T\right) \xrightarrow{d} \text{something}.$$

Def (descriptive).

$(B_t : t \geq 0)$ is a Brownian Motion (Wiener process)

1. $B_0 = 0$
2. For $t_1, t_2, \ldots, t_k$, $(B_{t_1}, B_{t_2}, \ldots B_{t_k})$ is multivariate normal.

(A process satisfying this is called Gaussian process)

3. For $t > s$, $B_t - B_s \sim \mathcal{N}(0, t-s)$ indp from the past $(B_r)_{0 \leq r \leq s}$.
4. $(B_t : t \geq 0)$ is cts function of t, a.s.

(Existence by Kolmogorov extension thm).

Fact.: $(B_t : t \geq 0)$ is Markov

($\forall t$, $(B_r : r > t) \perp (B_r : 0 \leq r \leq t) \mid B_t$.)

(Also applies to stopping time t : strong Markov)

$(B_t : t \geq 0)$ is a martingale.

- $\mathbb{E}|B_t| < +\infty$ $\forall t$
- $\mathbb{E}[B_t \mid \mathcal{F}_s] = B_s$ (for $s < t$).

Similarly, cts-time version of stopping time.

- A random time T s.t. $\forall t \geq 0$ the event $\{T \leq t\}$ determined by $\mathcal{F}_t$ (i.e. $(X_s)_{0 \leq s \leq t}$).

(Properties of discrete stopping times extend).

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cts-time OST.

- $(X_t)_{t \geq 0}$ cts-time MG (cts sample path).
- T is stopping time, $\mathbb{P}(T < +\infty) = 1$.

Suppose (i) $\mathbb{E}[|X_T|] < +\infty$ (ii) $\lim_{t \to +\infty} \mathbb{E}[|X_t| \cdot \mathbb{1}\{T \geq t\}] = 0$.

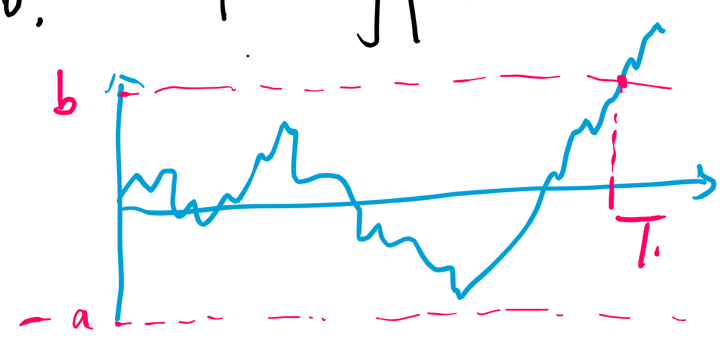
Then we have $\mathbb{E}[X_T] = \mathbb{E}[X_0]$.

(also holds under uniform integrability).

Application: cts-time gambler's ruin

$(B_t : t \geq 0)$ BM.

$a, b > 0$. $T = \inf\{t \geq 0 : B_t = -a \text{ or } B_t = b\}$.



$|B_t|$ is bounded by $a \vee b$ up to ~~time~~ T.

So OST applies

$$0 = \mathbb{E}[B_0] = \mathbb{E}[B_T] = b \cdot \mathbb{P}(B_T = b) - a\,\mathbb{P}(B_T = a).$$

So we can solve $\mathbb{P}(B_T = b) = \dfrac{a}{a+b}$.

Similarly, we can construct

$M_t = B_t^2 - t$. is a MG (easy to verify).

$$\mathbb{E}[|M_t| \cdot \mathbb{1}_{t \le T}].$$

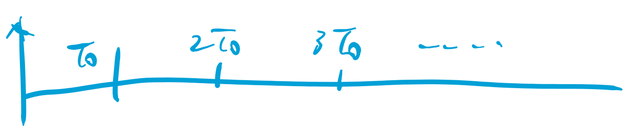
$$\le \mathbb{E}[(a^2 \vee b^2 + T) \cdot \mathbb{1}\{T \ge t\}].$$

To show that this goes to 0, we can show

$\exists\ c, T_0 > 0$. s.t. $\mathbb{P}(T \ge kT_0) \le (1-c)^k\ (\forall k)$.

So that $\mathbb{E}[T] < +\infty$, and therefore $\mathbb{E}[T\,\mathbb{1}\{T \ge t\}] \to 0$.

We can apply same argument as DTMC case.



$$\inf_{x \in (-a,b)} \mathbb{P}_x\left(\begin{array}{c}\text{reaching } -a \text{ or } b \\ \text{within time } T_0\end{array}\right) > 0.$$

So by OST.

$$0 = \mathbb{E}[M_0] = \mathbb{E}[M_T]$$

$$= \mathbb{E}[B_T^2] - \mathbb{E}[T]$$

$$= b^2 \cdot \frac{a}{a+b} + (-a)^2 \cdot \frac{b}{a+b} - \mathbb{E}[T]$$

So we conclude $\mathbb{E}[T] = ab$.

Remarks.

- We can also compute moments / Laplace transform / etc. $(\mathbb{E}[e^{-aT}])$
- There're more principled ways of constructing / verifying MG.

Reflection principle.

$(B_t : t \geq 0)$ BM.

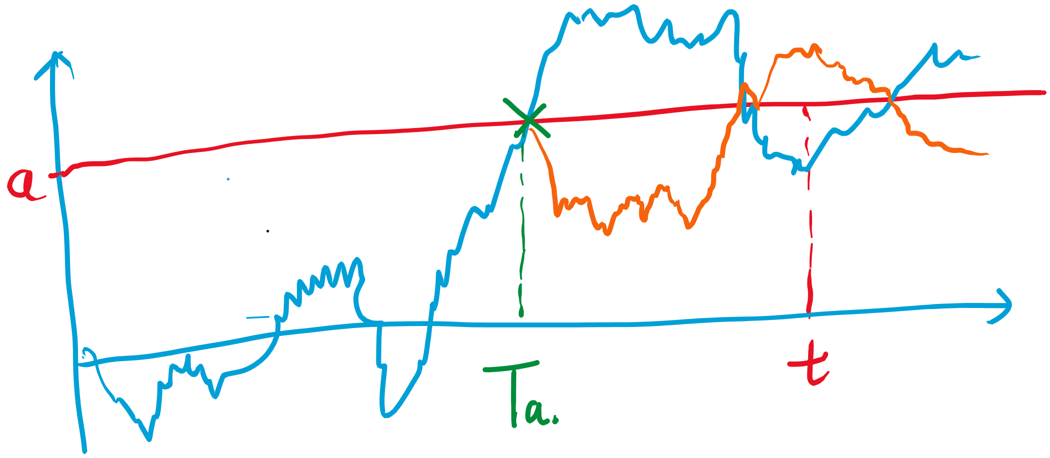
$$T_a = \inf\{t \geq 0 : B_t \geq a\}$$ for some $a > 0$

Question: distribution of T_a?

Need to compute $\mathbb{P}(T_a \leq t)$

for any $t > 0$.

$$\mathbb{P}(T_a \le t) = \mathbb{P}\left(\max_{0\le s\le t} B_s \geqslant a\right).$$



$$T_a \le t, \begin{cases} T_a \le t, \ B_t < a & \text{(case I)} \\ T_a \le t, \ B_t > a & \text{(case II)}. \end{cases}$$

($\mathbb{P}(B_t = a) = 0$. so we can ignore).

$$\mathbb{P}(\text{case II}) = \mathbb{P}(B_t > a) = \frac{1}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{+\infty} \exp\left(-\frac{s^2}{2}\right) ds$$

$$= \Phi\left(-\frac{a}{\sqrt{t}}\right).$$

$$\mathbb{P}(\text{case I}) = \mathbb{P}(\text{case II}).$$

A more rigorous argument.

$$\mathbb{P}(B_t \geq a) = \mathbb{P}(T_a \le t) \cdot \mathbb{P}(B_t \geq a \mid T_a \le t).$$

$$\Phi(-a/\sqrt{t})$$

of interests.

$$\mathbb{P}(B_t \geq a \mid T_a \leq t).$$

$$= \mathbb{P}(B_t - B_T \geq 0 \mid T_a \leq t).$$

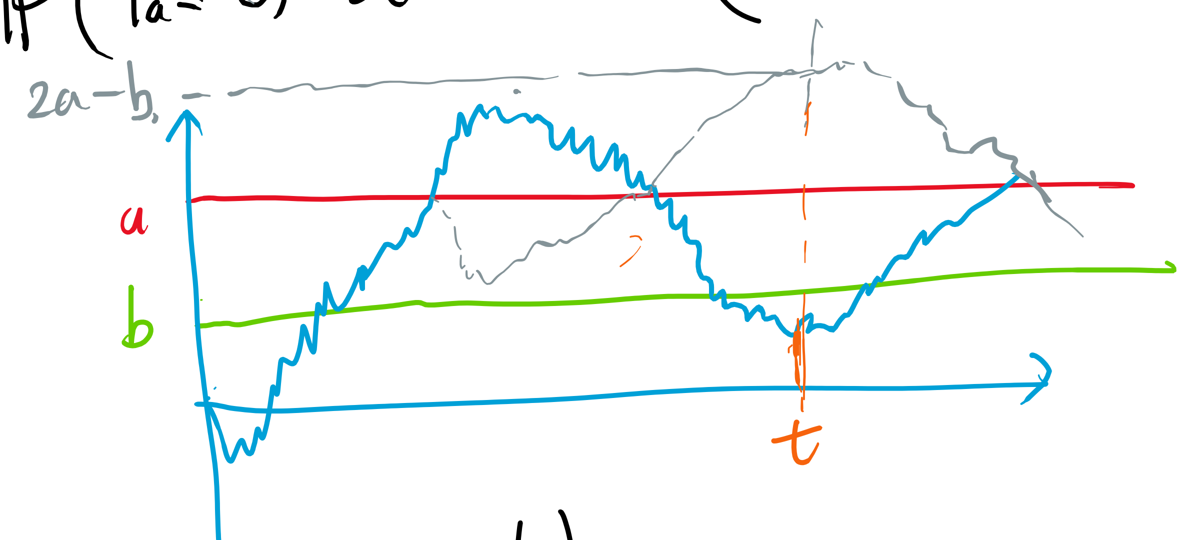
(By strong Markov property. $(B_t - B_T)_{t \geq T}$ is a fresh new BM)

$$= \frac{1}{2}.$$

So $\mathbb{P}(T_a \leq t) = 2\mathbb{P}(B_t \geq a) = 2\Phi(-a/\sqrt{t}).$

One more step. We can compute joint distribution of (T_a, B_t).

$\mathbb{P}(T_a \leq t,\ B_t \leq b)$ (for $b < a$)



$$\mathbb{P}(T_a \leq t,\ B_t \leq b)$$

$$= \mathbb{P}(T_a \leq t) \cdot \mathbb{P}(B_t \leq b \mid T_a \leq t)$$

$$= \mathbb{P}(T_a \leq t) \cdot \mathbb{P}(B_t - B_{T_a} \leq -(a-b) \mid T_a \leq t)$$

$$= \mathbb{P}(T_a \le t) \cdot \mathbb{P}(B_t - B_{T_a} \ge a - b \mid T_a \le t)$$

$$= \mathbb{P}(T_a \le t, \ B_t \ge 2a - b) = \mathbb{P}(B_t \ge 2a - b).$$

Corollary:

$(\forall a)$. $$\mathbb{P}(T_a \le t) = 2 \cdot \mathbb{P}(B_t \ge a) = \mathbb{P}(|B_t| \ge a)$$

$$\mathbb{P}(T_a \le t) = \mathbb{P}\left(\max_{0 \le s \le t} B_s \ge a\right)$$

So. $$\max_{0 \le s \le t} B_s \overset{d}{=} |B_t|.$$

Stochastic calculus.

Goal: make sense of "calculus" for BM.

Difficulty: $$\frac{B_{t+\Delta t} - B_t}{\Delta t} \sim \mathcal{N}\left(0, \frac{1}{\Delta t}\right).$$

$\Delta t \to 0$, var $\longrightarrow +\infty$.

$$\mathbb{P}(B \text{ is differentiable at } t) = 0.$$

Indeed, B is everywhere cts but nowhere differentiable (a.s.)

In practice, we want to compute

$$\int_0^t Y_s \, dB_s$$

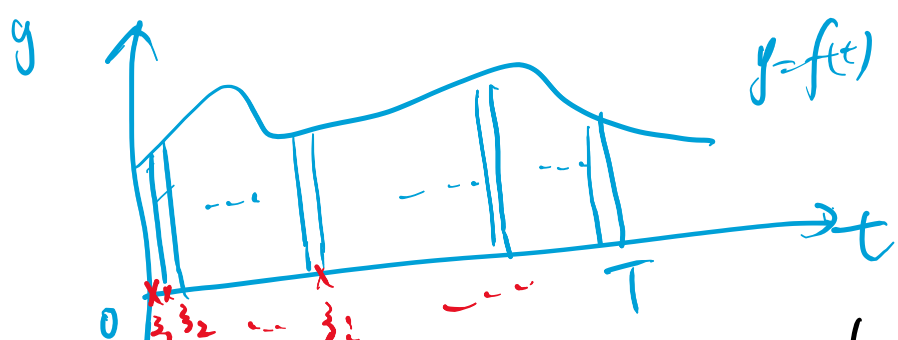
$(Y_t : t \geq 0)$ is a stochastic process (adapted to the filtration of BM)

i.e. Y_t is determined by $(B_s : 0 \leq s \leq t)$.

(Analogy: Stieltjes integral $\int_0^t f(s)\, dg(s)$ for g differentiable)

Recall: $$\int_0^T f(t)\, dt = \lim_{\Delta t \to 0} \sum_{i=1}^{N} f(\xi_i)\, \Delta t$$

Limit is indp of the choice of ξ_i's.



$$\int_0^T f(t)\, dg(t) = \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} f(\xi_i) \cdot \Big(g\big((i+1)\Delta t\big) - g(i\Delta t) \Big).$$

$$\int_0^T Y_s \, dB_s \overset{?}{=} \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} Y_{\xi_i} \left(B_{(i+1)\Delta t} - B_{i\Delta t} \right).$$

e.g. Let $Y_t = B_t$ $(\forall t)$. $\int_0^T B_t \, dB_t$.

$$\sum_{i=0}^{N-1} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot B_{\xi_i}$$

$$= \underbrace{\sum_{i=0}^{N-1} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right) \cdot B_{i\Delta t}}_{\text{Term 1}} + \underbrace{\sum_{i=0}^{N-1} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right)\left(B_{\xi_i} - B_{i\Delta t}\right)}_{\text{Term 2}}$$

Does Term 2 $\to 0$?

e.g. $\xi_i = \left(i + \frac{1}{2}\right) \Delta t$.

$$\text{Term 2} = \sum_{i=0}^{N-1} \left(B_{(i+1)\Delta t} - B_{i\Delta t}\right)\left(B_{(i+\frac{1}{2})\Delta t} - B_{i\Delta t}\right).$$

Each terms in the summation are i.i.d.

$$\mathbb{E}[\cdots\cdots] = \frac{\Delta t}{2}.$$

By SLLN. Term 2 $\xrightarrow{a.s.} \frac{T}{2} \neq 0$.

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there're different ways of choosing ξ_i's leading to different integrals.

Each has pros/cons.

(Analogous to betting strategy/DT MG.

$(X_t)_{t\geq 0}$ is MG.

$(Y_t)_{t\geq 0}$ is adapted (betting strategy)

$$M_t = \sum_{k\geq 0}^{t-1} Y_k (X_{k+1} - X_k) \quad \text{is a MG.}$$

(when X is SRW)

$$\mathbb{E}[M_t^2] = \sum_{k=0}^{t-1} \mathbb{E}[Y_k^2 \cdot (X_{k+1} - X_k)^2] = \sum_{k=0}^{t-1} \mathbb{E}[Y_k^2]$$

Here we choose left endpoints and get

- MG
- L^2 isometry property.

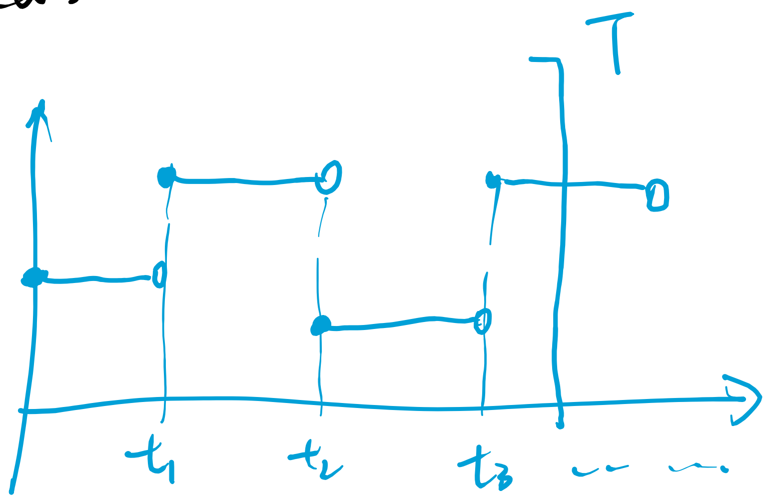
Want to extend to $\int_0^T Y_t \, dB_t$ by taking left endpts, preserving two properties.

Technical difficulty:

$$\sum_{i=0}^{N-1} Y_{i\Delta t} \cdot (B_{(i+1)\Delta t} - B_{i\Delta t}) \xrightarrow{?} \text{something.}$$

Roadmap:

— For piecewise constant Y



$t_1, t_2, \ldots$ deterministic times.

$$Y_s = Y^{(i)} \quad \text{for } s \in [t_i, t_{i+1}).$$

$$\int_0^T Y_t \, dB_t = \sum_{t_{i+1} \leq T} Y^{(i)} \cdot (B_{t_{i+1}} - B_{t_i}) + Y^{(i_0+1)} (B_T - B_{t_{i_0}})$$

where $i_0 := \max\{i : t_i \leq T\}$.

For piecewise const Y,

- $\left(\int_0^T Y_t \, dB_t\right)_{T \geq 0}$ is MG.
- $\mathbb{E}\left[\left|\int_0^T Y_t \, dB_t\right|^2\right] = \int_0^T \mathbb{E}[|Y_t|^2] \, dt.$

Extending to general adapted process.

$(Y_t)_{t\geq 0}$ cts sample path

adapted (Y_t only depends on $(B_s)_{0\leq s\leq t}$).

Piecewise constant approximation:

$\forall n \geq 1$, $Y_t^{(n)} = Y_{i\Delta t}$ for $t \in [i\Delta t, (i+1)\Delta t)$

where $\Delta t = T/n$.

Fact (real analysis): If $(Y_t)_{t\geq 0}$ is cts.

$$\lim_{n\to+\infty} \int_0^T \mathbb{E}\left[|Y_t - Y_t^{(n)}|^2\right] dt = 0.$$

(piecewise const functions are dense in the space of cts func under L^2).

$$\lim_{n,m\to+\infty} \int_0^T \mathbb{E}\left[|Y_t^{(m)} - Y_t^{(n)}|^2\right] = 0.$$

By isometry,

$$\mathbb{E}\left|\int_0^T Y_t^{(m)} dB_t - \int_0^T Y_t^{(n)} dB_t\right|^2 \longrightarrow 0$$

as $m, n \to +\infty$.

So $\left(\int_0^T Y_t^{(n)} dB_t\right)_{n\geq 0}$ forms a Cauchy seq.

in L^2. the limit exists.

We define $\int_0^T Y_t \, dB_t$ as the L^2 limit.

"Itô's integral".

$$\left\{\begin{array}{l} \left(\int_0^t Y_s \, dB_s\right)_{t\geq 0} \text{ is MG.} \\ \mathbb{E}\left(\int_0^t Y_s \, dB_s\right)^2 = \int_0^t \mathbb{E}[Y_s^2] \, ds \\ \int_0^t \left(a Y_s^{(1)} + b Y_s^{(2)}\right) dB_s = a \int_0^t Y_s^{(1)} dB_s + b \int_0^t Y_s^{(2)} dB_s \end{array}\right.$$