STA 447/2006 Lecture 9

Recall from last lecture

$$\int_0^T Y_t \, dB_t$$

- $(Y_t)_{t \geq 0}$ adapted, cts.

Idea: first define for piecewise const Y and then take L^2 limit.

Easily verified from construction.

$$\begin{cases} \left(\int_0^t Y_s \, dB_s\right)_{t\geq 0} \text{ is a martingale} \\ \mathbb{E}\left[\left(\int_0^t Y_s \, dB_s\right)^2\right] = \int_0^t \mathbb{E}[Y_s^2] \, ds \\ \int_0^t \left(a Y_s^{(1)} + b Y_s^{(2)}\right) dB_s = a \int_0^t Y_s^{(1)} \, dB_s + b \int_0^t Y_s^{(2)} \, dB_s. \end{cases}$$

In calculus, we have

$$\int_a^b f'(x) dx = f(b) - f(a).$$

Differential form $\quad df(x) = f'(x) \, dx.$

Similar eq. for Itô's integral?

Counter-example

$$f(x) = \frac{1}{2}x^2. \quad f'(x) = x.$$

$$\int_0^T B_t \, dB_t \overset{?}{=} f(B_T) - f(B_0) = \frac{1}{2}B_T^2$$

(If B were a cts differentiable func, this would be true).

This cannot be true:

- $\left(\int_0^T B_t dB_t\right)_{T \geq 0}$ is MG
- $E[f(B_T)] = \frac{T}{2} > 0.$

Thm. Suppose f is twice cts differentiable.

$$f(B_T) - f(B_0)$$

$$= \underbrace{\int_0^T f'(B_t) \, dB_t}_{\text{Itô integral}} + \underbrace{\frac{1}{2}\int_0^T f''(B_t) \, dt.}_{\text{Riemann integral.}}$$

(in differential form)

$$df(B_t) = f'(B_t) \, dB_t + \frac{1}{2} f''(B_t) \, dt.$$

e.g. $f(x) = \frac{1}{2}x^2$. $\quad f'(x) = x \quad f''(x) = 1$

$$\frac{1}{2}B_T^2 - \frac{1}{2}B_0^2 = \int_0^T B_t \, dB_t + \frac{1}{2}\int_0^T dt$$

So $\int_0^T B_t \, dB_t = \frac{1}{2}B_T^2 - \frac{1}{2}T$.

From last lecture, we have verified $(B_t^2 - t)_{t \geq 0}$ is MG.

Actually we can systematically construct MGs.

For any f (twice cts diff)

$$\left(f(B_t) - f(B_0) - \frac{1}{2}\int_0^t f''(B_s)\, ds\right)_{t \geq 0}$$

is an MG.

e.g. $f(x) = e^x$.

$$d f(B_t) = e^{B_t} dB_t + \frac{1}{2} e^{B_t} dt,$$

$$\left(e^{B_t} - \frac{1}{2}\int_0^t e^{B_s} ds\right)_{t \geq 0}$$ is MG.

Additional remark:

If you take $\xi_i = \frac{1}{2}(t_i + t_{i+1})$

in defining stochastic integral

You'll have

$$\int_0^T f'(B_t) \circ dB_t = f(B_T) - f(B_0).$$

But you lose MG property.

"Stratonovich integral".

Proof idea: take $0 \le t_0 < t_1 < \cdots < t_N = T$ $\left(t_{i+1} - t_i = \Delta t = \frac{T}{N}\right)$

$$f(B_T) - f(B_0) = \sum_{i=0}^{N-1} \left(f(B_{t_{i+1}}) - f(B_{t_i}) \right).$$

$f(B_{t_{i+1}}) - f(B_{t_i})$ (Taylor expansion)

$$= f'(B_{t_i}) \cdot (B_{t_{i+1}} - B_{t_i}) + \frac{1}{2} f''(B_{t_i}) \cdot (B_{t_{i+1}} - B_{t_i})^2.$$

$$+ o\left(|B_{t_{i+1}} - B_{t_i}|^2 \right) = o_p(\Delta t).$$

So we have

$$f(B_T) - f(B_0) = \underbrace{\sum_{i=0}^{N-1} f'(B_{t_i}) \cdot (B_{t_{i+1}} - B_{t_i})}_{\text{stoch. int.}} + \frac{1}{2} \sum_{i=0}^{N-1} f''(B_{t_i}) (B_{t_{i+1}} - B_{t_i})^2$$

$$+ o_p(N \cdot \Delta t) \longrightarrow 0 \quad \text{as } N \to +\infty.$$

Key step: quadratic term.

eg. $f(x) = \frac{1}{2}x^2$.

$$\frac{1}{2}\sum_{i=0}^{N-1}\left(B_{t_{i+1}} - B_{t_i}\right)^2 \xrightarrow{a.s.} \frac{T}{2}. \quad \text{(by SLLN)}.$$

(iid χ^2 random variables)

Extending to general functions.

$$\text{const} \overset{(i)}{\Longrightarrow} \text{piecewise const} \overset{(ii)}{\Longrightarrow} \text{cts}.$$

Consider $(g_t)_{t\geq 0}$ adapted stochastic process w/ cts path

Want: $$\lim_{N\to+\infty}\sum_{i=0}^{N-1} g_{t_i}\cdot\left(B_{t_{i+1}} - B_{t_i}\right)^2 = \int_0^T g_t\,dt.$$

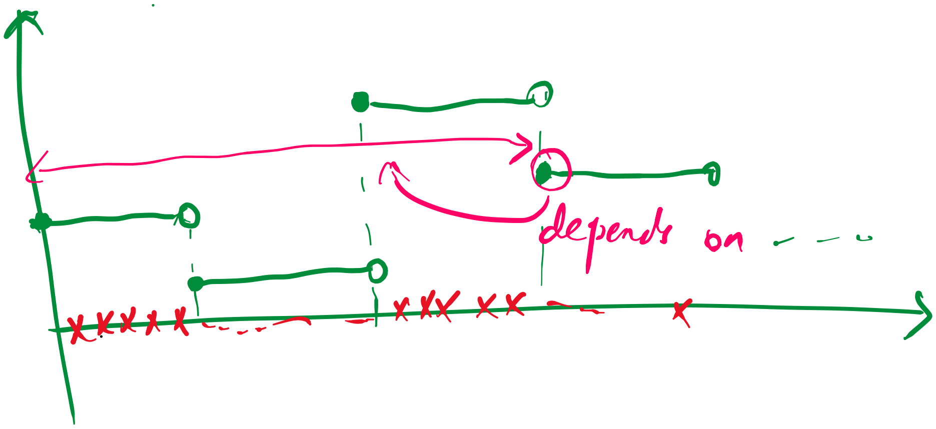
(Note: g_t itself can be random, and depending on the past $(B_s)_{0\leq s\leq t}$)

Step (i): const $\Longrightarrow$ piecewise const.

"Piecewise const process"

$$g_t = g_{\tau_k} \quad \text{for} \quad t\in[\tau_k, \tau_{k+1})$$

where $\tau_0, \tau_1, \tau_2, \cdots$ are deterministic, equispaced time steps.



"Different time-scale": Fix $\tau_0, \tau_1, \tau_2, \cdots, \tau_k, \cdots$

Let $N \to +\infty$, $\Delta t \to 0$.

We can condition on $\mathcal{F}_{\tau_k}$, g_{τ_k} is determined by the history.

$$g_t = g_{\tau_k} \text{ for } t \in [\tau_k, \tau_{k+1}).$$

Using SLLN arguments above.

$$\lim_{N \to +\infty} \sum_{\tau_k \le t_i < \tau_{k+1}} g_{t_i} \cdot (B_{t_{i+1}} - B_{t_i})^2 \xrightarrow{a.s.} \int_{\tau_k}^{\tau_{k+1}} g_t \, dt.$$

Step (ii). Piecewise constant $\Rightarrow$ cts.

Given $(g_t)_{t \ge 0}$ cts, adapted.

$(g_t^{(m)})_{t \ge 0}$: piecewise const approx to $(g_t)_{t \ge 0}$ by dividing $[0, T]$ into m parts.

$$\sup_{0\le t\le T}\left|g_t - g_t^{(m)}\right| \to 0 \quad (\text{as } m\to+\infty).$$

So we have

$$\lim_{m\to+\infty}\left|\int_0^T g_t\,dt - \int_0^T g_t^{(m)}\,dt\right| = 0.$$

Putting them together, we proved Itô's thm

Extensions of Itô's formula.

Ideal: most general form $\left(Z_t = Z_0 + \int_0^t X_s\,ds + \int_0^t Y_s\,dB_s\right)$.

$$dZ_t = X_t\,dt + Y_t\,dB_t.$$

Want to compute $df(t, Z_t)$

using X_t, Y_t

(Generally. we can consider $X_t, Z_t \in \mathbb{R}^d$, $Y_t \in \mathbb{R}^{d\times d}$).

Special cases:

- $f(Z_t)$
- $f(t, B_t)$ $(X_t = 0,\ Y_t = 1)$.

$$f(Z_T)-f(Z_0)$$

$$=\sum_{i=0}^{N-1}\left(f(Z_{t_{i+1}})-f(Z_{t_i})\right)$$

$$f(Z_{t_{i+1}})-f(Z_{t_i})=\underline{f'(Z_{t_i})\cdot(Z_{t_{i+1}}-Z_{t_i})}\ ①$$

$$+\underline{\tfrac{1}{2}f''(Z_{t_i})\left(Z_{t_{i+1}}-Z_{t_i}\right)^2}\ ②$$

$$+\ o\left(\left|Z_{t_{i+1}}-Z_{t_i}\right|^2\right).$$

$o_p(\Delta t)$.

The summation of this term $\to 0$. (a.s.).

For term ①: $\quad dZ_t = X_t\,dt + Y_t\,dB_t$

$$f'(Z_{t_i})\cdot(Z_{t_{i+1}}-Z_{t_i})$$

$$=f'(Z_{t_i})\cdot\left\{\int_{t_i}^{t_{i+1}}X_t\,dt+\int_{t_i}^{t_{i+1}}Y_t\,dB_t\right\}.$$

$$\sum_{i=0}^{N-1}f'(Z_{t_i})\int_{t_i}^{t_{i+1}}X_t\,dt\xrightarrow{a.s.}\int_0^T f'(Z_t)\cdot X_t\,dt$$

(as $N\to+\infty$).

(similar to Riemann-Stieltjes integral).

$$\sum_{i=0}^{N-1} f'(Z_{t_i}) \int_{t_i}^{t_{i+1}} Y_t \, dB_t$$

$$= \int_0^T g_t^{(N)} \cdot Y_t \, dB_t$$

$(g_t^{(N)})_{t\geq 0}$ is piecewise const approx to $(f'(Z_t))_{t\geq 0}$

Following argument in the definition of Itô integral,

$$\int_0^T g_t^{(N)} Y_t \, dB_t \xrightarrow{L^2} \int_0^T f'(Z_t) Y_t \, dB_t.$$

Term 2.

$$\frac{1}{2} \lim_{N\to+\infty} \sum_{i=0}^{N-1} f''(Z_{t_i}) \cdot (Z_{t_{i+1}} - Z_{t_i})^2$$

To deal with this process, we introduce:

Def. Given a process $(Z_t)_{t\geq 0}$. we define its quadratic variation process:

$$\langle Z \rangle_t = \lim_{N\to+\infty} \sum_{i=0}^{N-1} (Z_{t_{i+1}} - Z_{t_i})^2$$

where $t_i = \frac{t}{N}$. (equispaced).

(c.f. total variation $\lim_{N\to+\infty} \sum_{i=0}^{N-1} \left| Z_{t_{i+1}} - Z_{t_i} \right|$.)

- $\langle Z \rangle_t \geq \langle Z \rangle_s \geq 0$ for $0 < s < t$.
- $(Z_t)_{t\geq 0}$ cts $\Rightarrow$ $\langle Z \rangle$ cts.

For term ② we use piecewise const approximation to show that

$$\lim_{N\to+\infty} \sum_{i=0}^{N-1} f''(Z_{t_i}) \cdot \left(Z_{t_{i+1}} - Z_{t_i}\right)^2 \xrightarrow{a.s.} \underline{\int_0^T f''(Z_t)\, d\langle Z\rangle_t}.$$

$\langle Z \rangle_t$ non-dec in t.

Riemann-Stieltjes integral.

Fact. If $dZ_t = X_t\,dt + Y_t\,dB_t$

then $d\langle Z\rangle_t = Y_t^2\,dt$.

Proof of the fact.

$$\left(Z_{t_{i+1}} - Z_{t_i}\right)^2 = \Big[\underbrace{\int_{t_i}^{t_{i+1}} X_t\,dt}_{O(\Delta t)} + \underbrace{\int_{t_i}^{t_{i+1}} Y_t\,dB_t}_{O(\sqrt{\Delta t})}\Big]^2.$$

$$= \left(\int_{t_i}^{t_{i+1}} Y_t \, dB_t\right)^2 + \underbrace{O(|\Delta t|^2) + O\left(|\Delta t|^{3/2}\right)}$$

$$\Sigma(\cdots) \leq O\left(N \cdot |\Delta t|^{\frac{3}{2}}\right) \to 0.$$

$$\sum_{i=0}^{N-1} \left(Z_{t_{i+1}} - Z_{t_i}\right)^2$$

$$= \underbrace{\sum_{i=0}^{N-1} \left(\int_{t_i}^{t_{i+1}} Y_t \, dB_t\right)^2} + o_p(1) \to 0.$$

Conditionally on $\mathcal{F}_{t_i}$.

$$\mathbb{E}\left[\left(\int_{t_i}^{t_{i+1}} Y_t \, dB_t\right)^2 \middle| \mathcal{F}_{t_i}\right] = \int_{t_i}^{t_{i+1}} \mathbb{E}[Y_t^2] \, dt.$$

(martingale difference sum: can bound deviation)

Using piecewise const approximation etc

$$\lim_{N \to +\infty} \sum_{i=0}^{N-1} \left(\int_{t_i}^{t_{i+1}} Y_t \, dB_t\right)^2 = \int_0^T Y_t^2 \, dt.$$

Conclusion:

$$df(Z_t) = f'(Z_t)\, dZ_t + \frac{1}{2} f''(Z_t) \cdot d\langle Z \rangle_t$$

$$= \left(f'(Z_t) X_t + \frac{1}{2} f''(Z_t) \cdot Y_t^2\right) dt + f'(Z_t) \cdot Y_t \, dB_t$$

Similarly, time-inhomogeneous version.

$$f(t, Z_t) - f(0, Z_0)$$

$$= \sum_{i=0}^{N-1} \left(f(t_{i+1}, Z_{t_{i+1}}) - f(t_i, Z_{t_i}) \right)$$

$$\left(f(t_{i+1}, Z_{t_{i+1}}) - f(t_i, Z_{t_{i+1}}) \right) + \left(f(t_i, Z_{t_{i+1}}) - f(t_i, Z_{t_i}) \right)$$

$$\left(f(t_{i+1}, Z_{t_{i+1}}) - f(t_i, Z_{t_{i+1}}) \right) \approx \frac{\partial f}{\partial t}(t_i, Z_{t_{i+1}}) \cdot \Delta t$$

$$\left(f(t_i, Z_{t_{i+1}}) - f(t_i, Z_{t_i}) \right) \approx \frac{\partial f}{\partial z}(t_i, Z_{t_i}) \cdot (Z_{t_{i+1}} - Z_{t_i}) + \frac{1}{2} \frac{\partial^2 f}{\partial z^2}(t_i, Z_{t_i}) (Z_{t_{i+1}} - Z_{t_i})^2$$

Putting them together,

$$df(t, Z_t) = \frac{\partial f}{\partial t}(t, Z_t)\, dt + \frac{\partial f}{\partial z}(t, Z_t)\, dZ_t + \frac{1}{2} \cdot \frac{\partial^2 f}{\partial z^2}(t, Z_t) \cdot d\langle Z \rangle_t.$$

eg. $f(t, z) = \exp(at + bz)$

$$df(t, B_t) = a \cdot \exp(at + bB_t)\,dt + b \cdot \exp(at + bB_t)\,dB_t + \frac{b^2}{2} \exp(at + bB_t)\,dt.$$

$$= \left(a + \frac{b^2}{2}\right) f(t, B_t)\,dt + b \cdot f(t, B_t)\,dB_t.$$

Special case: $a + \frac{b^2}{2} = 0.$

$\forall b$, $\left(\exp\left(bB_t - \frac{b^2}{2}t\right)\right)_{t \geq 0}$ is MG.

(No need to compute expectations, etc.)

(Verify MG by taking Itô expansion).

Product rule.

In calculus, we have

$$d(fg) = f \cdot dg + g \cdot df.$$

For stochastic calculus.

$$dZ_t^{(i)} = X_t^{(i)} dt + Y_t^{(i)} dB_t. \qquad i \in \{1, 2\}.$$

$$d\left[(Z_t)^2\right] = 2Z_t\, dZ_t + d\langle Z \rangle_t.$$

$$= 2Z_t X_t\, dt + 2 Z_t Y_t\, dB_t + 2 Y_t^2\, dt.$$

How about

$$d\left(Z_t^{(1)} Z_t^{(2)}\right)$$

(polarization trick)

$$= d\left[\left(\frac{Z_t^{(1)} + Z_t^{(2)}}{2}\right)^2\right] - d\left[\left(\frac{Z_t^{(1)} - Z_t^{(2)}}{2}\right)^2\right]$$

$$= Z_t^{(1)} dZ_t^{(2)} + Z_t^{(2)} dZ_t^{(1)} + Y_t^{(1)} \cdot Y_t^{(2)}\, dt.$$

In general, we can define cross variation

$$\langle Z^{(1)}, Z^{(2)} \rangle_t = \lim_{N \to +\infty} \sum_{i=0}^{N-1} \left(Z_{t_{i+1}}^{(1)} - Z_{t_i}^{(1)}\right)\left(Z_{t_{i+1}}^{(2)} - Z_{t_i}^{(2)}\right)$$

$$d\langle Z^{(1)}, Z^{(2)} \rangle_t = Y_t^{(1)} Y_t^{(2)}\, dt.$$